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Soliton–cuspon interaction for the Camassa–Holm equation

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Abstract. The interaction of different kinds of solitary waves of the Camassa–Holm equation is investigated. We consider soliton–soliton, soliton–cuspon and cuspon–cuspon interactions. The description of these solutions had previously been shown to be reducible to the solution of an algebraic equation. Here we give explicit examples, numerically solving these algebraic equations and plotting the corresponding solutions. Further, we show that the interaction is elastic and leads to a shift in the position of the solitons or cuspons. We give the analytical expressions for this shift and represent graphically the coupled soliton–cuspon, soliton–soliton and cuspon–cuspon interactions.

1. Introduction

In this paper we will be interested in the study of the interaction of different kinds of travelling-wave solutions to the Camassa–Holm (CH) equation, which reads

$$u_t - u_{xxt} - 4u_x + uu_{xxx} - 3uu_x + 2u_x u_{xx} = 0. \tag{1}$$

This equation was obtained in [1] to describe the dynamics of long waves in shallow water, including higher-order corrections. Although it has been shown to be a completely integrable equation and to possess several kinds of interesting solutions [1–7], many aspects of its integrability are still under consideration. For example, the initial value problem has not been solved completely, although the particular case with a periodic initial condition was investigated in [8, 9].

Among particular solutions of equation (1) there are soliton solutions and cuspon solutions. Both are localized travelling waves. The study of the interaction of these waves and, in particular, the interaction of different kinds of travelling waves, is, naturally, a matter of interest.

As an integrable equation, explicit solutions to equation (1) may be constructed. In [6, 7] a class of (multi)-solitary wave solutions was obtained by using the $\bar{\partial}$-problem, encompassing solutions containing solitons and cuspons among others. These solutions are related to the roots of an algebraic equation, allowing us to reduce the construction of this kind of solution to the problem of calculating the roots of the above-mentioned algebraic equation. However, only in the simplest case can the roots be found explicitly. For instance, in the case where this equation is polynomial and of third degree, the corresponding solutions are single solitary waves. They are given in [6, 7]. The multi-soliton–cuspon solutions are related to the solutions of algebraic equations of degree $P > 4$. Their construction must involve numerical calculations.

These solutions are considered here. We restrict ourselves to two-wave interactions and consider numerically soliton–cuspon, soliton–soliton and cuspon–cuspon interactions. Note that we do not solve the CH equations directly by numerical methods. Rather, we proceed as far as possible with analytical methods and use numerical computations only in the final stage, to solve the algebraic equations and reconstruct the exact solutions. We find that the interaction is elastic in all cases and leads to a shift in the position of the soliton or cuspon. We derive analytical expressions for the shift after any number of soliton–cuspon collisions. They are compared with the numerical calculations for a coupled interaction. The underlying structure that ensures elasticity even between different kinds of solitary waves is the fact that these solutions may be obtained from the soliton solutions of another integrable equation (the deformed Sine–Gordon equation [7]).

2. Soliton–cuspon interaction

The N-soliton–cuspon solution of equation (1) may be represented by the system of equations

$$\psi(0) = -1 + \sum_{k=1}^{N} \frac{1}{1-b_k}\psi(b_k)\chi^{\alpha_k}\exp(\alpha_k\eta_k) \tag{2}$$

$$\psi(b_j) = \frac{1}{b_j - 1} + \sum_{k=1}^{N} \frac{1}{1-b_k-b_j}\psi(b_k)\chi^{\alpha_k}\exp(\alpha_k\eta_k) \tag{3}$$

$$\begin{gathered}\psi(0) = k\chi \qquad \chi = \exp(\Phi(x,t)) \qquad u = \frac{\Phi_t}{\Phi_x} \\ \eta_k = -\alpha_k(x - x_{0k} - v_k t) \qquad v_k = \beta_k/\alpha_k \\ \alpha_k = 2b_k - 1 \qquad \beta_k = \frac{1}{b_k} + \frac{1}{b_k - 1}\end{gathered} \tag{4}$$

where x_0 is the initial position and the constants b_k should satisfy the conditions

$$b_k \neq \tfrac{1}{2}, 0, 1 \qquad b_k + b_j \neq 1 \qquad b_k \neq b_j \qquad k \neq j.$$

Note that the system of equations (2), (3) is an algebraic one. For the sake of completeness and clarity, let us briefly discuss the one-soliton (cuspon) case, obtained by putting $N = 1$ into equations (2), (3). We write down the expressions for the amplitudes ($u_{\mathrm{cusp}}, u_{\mathrm{sol}}$) and velocities ($v_{\mathrm{cusp}}, v_{\mathrm{sol}}$) of the cuspon and the soliton and corresponding intervals of the parameter b_1, which determine whether the solution is a cuspon or a soliton:

$$u_{\mathrm{cusp}} = \frac{(2b_1-1)^2}{b_1(1-b_1)} \qquad v_{\mathrm{cusp}} = -\frac{1}{b_1(b_1-1)} < 0 \qquad b_1 < 0 \qquad b_1 > 1 \tag{5}$$

$$u_{\mathrm{sol}} = \frac{1}{b_1(1-b_1)} \qquad v_{\mathrm{sol}} = -\frac{1}{b_1(b_1-1)} > 0 \qquad 0 < b_1 < 1. \tag{6}$$

One can see that the amplitude and velocity do not change if we replace b_1 by $1 - b_1$. For this reason in the following we will consider only b_k in the interval $\frac{1}{2} < b_k < +\infty$.

In figure 1 we plot several examples of soliton and cuspon solutions, obtained by solving the above set of equations. To find the roots of these algebraic equations we use the Laguerre method [10,11] or the bracketing and bisection method [10] depending on whether the algebraic equation is polynomial or transcendental one.

Let us now return to the general multi-soliton case. The algebraic system (2)–(4) can be reduced to the algebraic equation of the form

$$\sum_{k=0}^{N} \sum_{j_1<j_2<\cdots<j_k<N} A_{j_1 j_2 \ldots j_k}(\chi)\exp(-\alpha_{j_1}\eta_{j_1} - \alpha_{j_2}\eta_{j_2} - \cdots - \alpha_{j_k}\eta_{j_k}) = 0 \qquad \eta_0 = 0 \tag{7}$$

where $A_{j_1 j_2 \ldots j_k}$ are functions of χ whose highest degree is $(1 + \alpha_{j_1} + \alpha_{j_2} + \cdots + \alpha_{j_k})$. Even in the $n = 2$ case, we have to numerically solve this algebraic equation, which is of no great

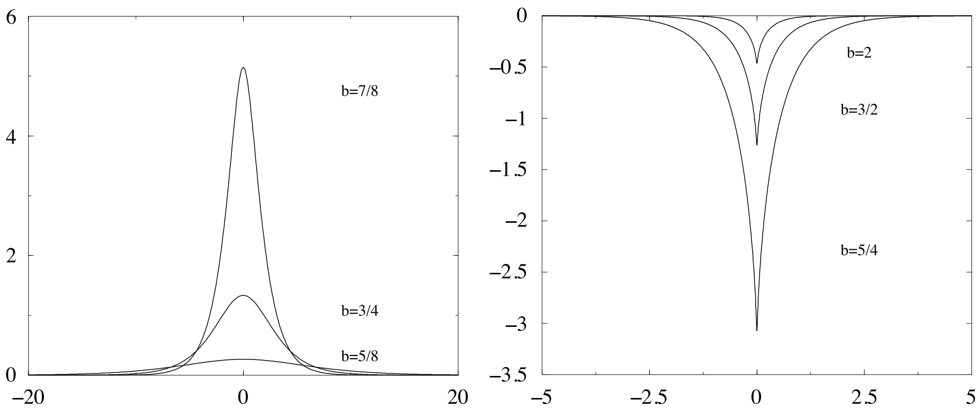

Figure 1. Soliton and cuspon solutions of the CH equation.

difficulty. However, it is not evident whether the algebraic equation (7) has *real* or *imaginary* solutions. We will be only interested in the solutions that generate real solutions to the CH equation.

In figures 2–4 we plot the soliton–soliton, soliton–cuspon and cuspon–cuspon solutions and give the related shifts. To do this we find the function χ by solving the equation (7) numerically with $N = 2$ and different values of parameters b_1 and b_2: $(b_1, b_2) = (\frac{3}{4}, \frac{5}{8})$, $(b_1, b_2) = (\frac{3}{4}, \frac{5}{4})$ and $(b_1, b_2) = (\frac{3}{2}, \frac{5}{4})$, for soliton–soliton, soliton–cuspon and cuspon–cuspon interactions, respectively. After this, the solution u of CH, equation (1), is obtained by $u = \chi_t/\chi_y$, which follows immediately from equation (4). In numerical calculations we use the *C* program implementation of Laguerre's method [10, 11] to find the roots of the polynomial equation. We choose only the roots which lead to the real bounded solutions of CH. For convenience, we plot the graphics in the frame moving with the velocity of the noninteracting slow soliton (cuspon). The horizontal axis represents $\eta = x - v_{\min}t$, where $v_{\min} = \min(v_1, v_2)$ and $v_k = 1/(b_k(b_k - 1))$, so that the slow soliton (cuspon) is centred at $\eta = 0$ before or after interaction.

3. Analytical results

We now reconsider equation (7) and extract analytical expressions for the shifts between the solitons (cuspons). Let us consider the situation when all solitons and cuspons are separated from one another at the initial moment ($t = 0$) so that the interaction between them is negligible. This can be done by choosing the constants x_{0k} in an appropriate way: $x_{01} > x_{02} > \cdots > x_{0N}$. Let us first consider the soliton (cuspon) furthest to the right. Recall that solitons and cuspons are moving to the left and right, respectively, so that a soliton from the right can interact only with solitons with smaller velocity and with cuspons. At the point where the right soliton (cuspon) (index 1) is situated one has

$$\eta_1 \sim 0 \qquad -\alpha_k\eta_k \ll -1 \qquad k \neq 1$$

so that the leading order in equation (7) gives us

$$A_0(\chi) + A_1(\chi)\exp(-\alpha_1\eta_1) = 0. \tag{8}$$

This is the usual solitary wave solution, examined in [7]. We represent this equation in the following form:

$$\boldsymbol{S_1} \equiv S(\chi, \eta_1) = (1 - 2b_1)(b_1 - 1)^2(1 - k\chi) - \exp(-\alpha_1\eta_1)\chi^{\alpha_1}(b_1^2 + k\chi(b_1 - 1)^2) = 0.$$

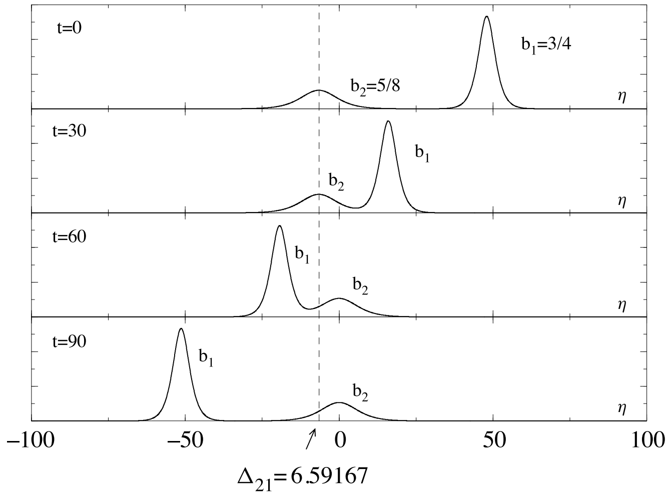

Figure 2. Soliton–soliton interaction.

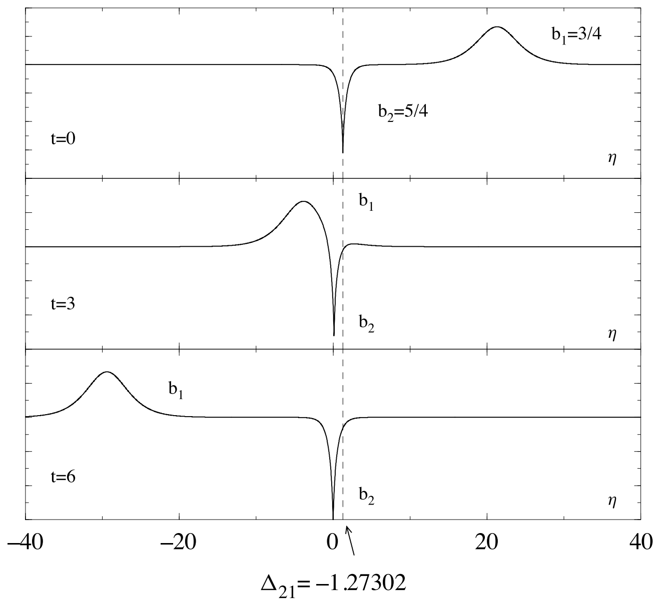

Figure 3. Soliton–cuspon interaction.

(9)

Consider the interaction with the first wave. This occurs when the another soliton (cuspon) (index 2) collides with the first one, goes through it and moves for a long enough distance for interaction to be negligible. In this position one has

$$\eta_1 \sim 0 \qquad -\alpha_2\eta_2 \gg 1 \qquad -\alpha_k\eta_k \ll -1 \qquad k \neq 1, 2$$

and the leading term becomes of the general form

$$\boldsymbol{S_2} \equiv \exp(\alpha_2\eta_2) f_2(b_1, b_2)\chi^{\alpha_2} S(\chi_1, \eta_1 - \triangle_1) = 0 \tag{10}$$

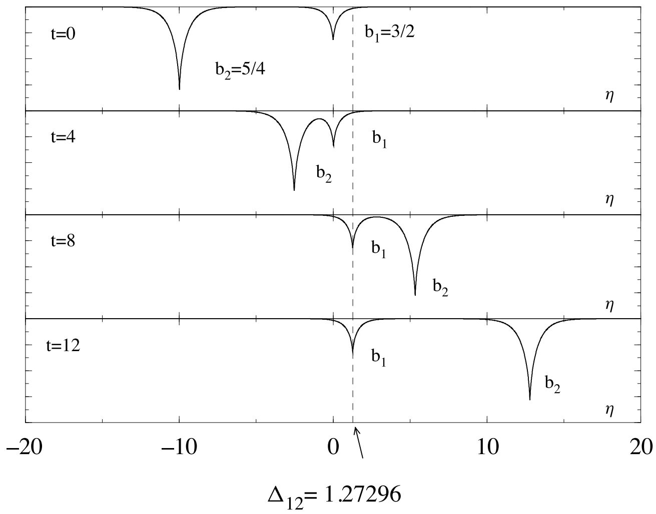

Figure 4. Cuspon–cuspon interaction.

with

$$\chi_1 = q(b_2)\chi \qquad q(b_2) = \left(\frac{1-b_2}{b_2}\right)^2 \tag{11}$$

$$\triangle_{12} = -\frac{2}{\alpha_1}\ln\left|\frac{b_1+b_2-1}{b_1-b_2}\left(\frac{1-b_2}{b_2}\right)^{\alpha_1}\right|. \tag{12}$$

After the interaction the second soliton (cuspon) has the shift

$$\triangle_{21} = \frac{2}{\alpha_2}\ln\left|\frac{b_1+b_2-1}{b_2-b_1}\left(\frac{1-b_1}{b_1}\right)^{\alpha_2}\right|. \tag{13}$$

The above expressions for the shifts are analogous to ones obtained in [2] for soliton–soliton interaction.

Notice that equation (10) is simply equation (9) with the replacement

$$\chi \to \chi_1, \eta_1 \to \eta_1 - \triangle_1.$$

The kth solitary wave CH equation after the interaction with $(k+1)$th has the same shape as the original solitary wave with the shift $\triangle_{k,k+1}$ in x, so that its position is $\tilde{x}_{0k} = x_{0k} - \triangle_{k,k+1}$, where x_{0k} is its position before the interaction. Analogously, after interacting with the Kth solitary wave, the leading term at the point $\eta_1 \sim 0$ will be

$$\boldsymbol{S_K} \equiv \exp(-\alpha_2\eta_2 - \cdots - \alpha_K\eta_K) f_K(b_1,\ldots,b_K)\chi^{\alpha_2+\ldots+\alpha_K}(S(\chi_K, \eta_1 - \tilde{\triangle}_{1K}) = 0$$

with

$$\chi_K = q(b_2,\ldots,b_K)\chi \qquad q(b_2,\ldots,b_K) = \prod_{m=2}^{K}\left(\frac{1-b_m}{b_m}\right)^2 \qquad \tilde{\triangle}_{1K} = \sum_{m=1}^{K}\triangle_{1m} \tag{14}$$

$$\triangle_{1m} = -\frac{2}{\alpha_1}\ln\left|\frac{b_1+b_m-1}{b_1-b_m}\left(\frac{1-b_m}{b_m}\right)^{\alpha_1}\right|. \tag{15}$$

The solitary wave has the original shape with shift $\triangle_K$ in its position: $\phi_K = \phi_0 - \tilde{\triangle}_K$. Furthermore, notice that equation (15) for the shift $\triangle_{1m}$ does not depend on the type of solitary waves (solitons or cuspons) under consideration.

Finally, let us present a brief analysis of the expression for the shift (12). There are five resonant points:

$$b_1 \to b_2 \qquad b_1 \to 1 - b_2 \tag{16}$$

$$b_2 \to 1 \qquad b_2 \to 0 \qquad b_1 \to \tfrac{1}{2}. \tag{17}$$

Equation (16) shows that the maximal interaction (shift) occurs for the waves moving with a small difference in velocity. The resonance of this kind occurs only for soliton–soliton or cuspon–cuspon interactions, which follows from equations (5), (6). About a soliton–cuspon interaction, we can only say that the shift of the slow wave is larger than that of the fast wave. Three other resonances (equation (17)) reflect the fact that when two solitons (cuspons) interact the shift of the slow one is larger than the shift of the fast one.

Acknowledgments

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